

# Statistics of Jamming in the discharge of a 2-D Silo

Gabriel Pérez<sup>\*1</sup>

<sup>1</sup>*Departamento de Física Aplicada, Centro de Investigación y de Estudios Avanzados  
del Instituto Politécnico Nacional, Unidad Mérida  
Apartado Postal 73 “Cordemex”, 97310 Mérida, Yucatán, México*

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Jamming and avalanche statistics are studied in a simulation of the discharge of a polydisperse ensemble of disks from a 2-D silo. Exponential distributions are found for the avalanche sizes for all sizes of the exit opening, in agreement with reported experiments. The average avalanche size grows quite fast with the size of the exit opening. Data for this growth agree better with a critical divergence with a large critical exponent, as reported for 3-D experiments, than with the exponential growth reported for 2-D experiments.

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## I. INTRODUCTION

The discharge of granular matter from a container is one of the most common phenomena of everyday life, but it is not yet completely understood. Be it the pouring of salt from a salt-shaker, of corn from a hopper, or of gravel from a truck, we count only with a few empirical rules to explain the process [1, 2]. Due to both its intrinsic interest and its obvious practical applications this phenomenon has received the attention of many researchers in recent times. It also provides a paradigmatic example of *jamming* [3], a recently proposed transition to a particular state of matter (also called *fragile matter*) that can support some stresses, called *compatible*, but flow under *incompatible* ones. Jammed systems are then non-solids that, under the influence of external forcing, develop some particular structures that block their flow [4].

A particularly simple and common example of granular matter discharge is given in hoppers and silos [5]. Three states of flow have been identified in these systems: dilute (gas-like), dense (liquid-like) and jammed (static, solid-like) [6, 7]. The transition from dilute to dense, as a function of the exit opening, is discontinuous, and shows hysteresis [7]. In these containers jamming is known to appear as soon as the size of the exit hole is reduced to a few times the average diameter of the particles inside. Experimental work in both 2-D and 3-D hoppers [8, 9] and silos [4, 6, 7, 9, 10] have shown that jamming depends only on the ratio between particle and exit hole sizes, as long as the diameter and height of the silo are large enough, that is, the silo is in its “thermodynamic limit”.

Among the many questions still remaining in the dynamics of silos, one of the simplest and more fundamental is the possible existence of a critical hole size such that for larger holes the flow cannot jam. The existence of such a size —actually, of a critical value for the ratio  $R = \text{exit-hole-size}/\text{grain-size}$ — has been shown in experiments for several types of granular media, including smooth and rough spheres, rice grains and lentils [10]. This is a somewhat puzzling result, since the same work shows that the distribution of avalanche sizes can be very well fitted to an exponential, and this in turn is consistent with a simple model where each grain —maybe cluster of grains— has a given probability of exiting the silo, uncorrelated to the behavior of other grains (clusters). Accepting this model, it becomes difficult to understand how a fully uncorrelated process can give rise to the long distance correlations one usually associates to criticality. It should also be mentioned that the experiments carried in [8, 9] also support the hypothesis of a fixed probability of exiting the silo for each grain, but point towards a probability of jamming that is exponentially decaying on  $R$ ; however, Ref. [9] also argues that it may be possible to fit its experimental data to either one of the two behaviors.

In this work we have carried out a simulation of a 2-D silo with variable hole size, with the intention of getting some information on the statistics of its discharge, and on the possible presence of criticality in this process. Even though simulational approaches cannot reproduce completely the dynamics of real experiments, they do give very good approximations to real flows, and should be able to find signatures of critical behavior, if present. They have the added advantage of allowing for continuous and unconstrained adjustments in the main parameters of the flow. There are some obvious limitations to this particular simulation that puts it at a disadvantage with respect to actual experiments, chiefly the fact that tracking the very long avalanches that appear for large exit hole sizes consume an inordinate amount of computer time. Still, a systematic study of the avalanches for different exit hole sizes allows the identification of a well defined trend.

## II. SIMULATION

The simulations reported here were done over an ensemble of  $N$  polydisperse disks, with diameter given by  $d_i = d_{ave} + x\Delta d$ , where  $d_{ave}$  is the average diameter,  $\Delta d$  is its maximum fluctuation, and  $x$  is randomly chosen from the  $[-1, 1]$  uniform distribution. The silo has a bottom size  $D$  and indefinite height. At the center of the bottom there is a hole of size  $d_H$ . The disks have a 2-D mass density  $\sigma$ , and the gravitational acceleration  $g$  acts in the negative  $z$  direction. Upon contact, the disks interact with the (perfectly rigid) walls of the silo, and with each other, via a linear spring with a constant that on loading has the value  $\kappa$ , and on unloading is reduced by a restitution factor  $\epsilon$ . This is an implementation of the *linear spring-dashpot model* [11, 12], using the two-couplings approach given in [13]. This approach is commonly used because of its robustness and simplicity. The interaction is complemented by dynamic and static friction, using for both the same coefficient  $\mu$ . For the static part the tangential spring model given in [11] is used, following the specific formulation of [12], with the corrections given in [14]. Equations of motion were integrated using a velocity-Verlet algorithm [15].

The only fixed numerical input in the problem is given by the gravitational constant; all quantities can be scaled, say to natural units where the disks' average diameter and mass are set to one. In this work we have preferred to implement a numerical experiment with standard units, and have used  $N = 2000$  disks with  $d_{ave} = 0.5$ ,  $\Delta d = 0.05$ ,  $D = 15$ ,  $\sigma = 0.8$  and  $\kappa = 4 \times 10^6$ , with all quantities given in the cgs system. This value for  $\kappa$  is not as large as could be expected for some hard real systems (steel or glass spheres, for instance), but it allows for a more efficient use of computer time. It can be realistic enough for softer grains, like the rice or lentils used in [10]. It has also been reported that changes of up one or two orders of magnitude in the stiffness of the grains have little influence on the results of these types of simulations [16]. For the adimensional quantities  $\mu$  and  $\epsilon$  we have set 0.5 and 0.9 respectively. Gravity is fixed as  $g = 981$ . The ratio between the diameters of the disks and the silo gives  $D/d_{ave} = 30$ , and for the number of disks used the silo gets filled up to a height of around 2.5 times  $D$ . These two values are large enough to put the silo in the thermodynamic limit [4, 10, 17], at least for the 3-D case. A time-step of 0.01 times the disk-disk collision time  $t_{coll}$ , which for the linear interaction used here is given by

$$t_{coll} = \pi \sqrt{\frac{m}{\kappa}},$$

neglecting a weak dependence on  $\epsilon$ . For later convenience, we also define the time and speed scales  $t_0 = \sqrt{d_{ave}/g}$  and  $v_0 = g t_0$ .

The simulation proceeds as follows: first the silo is closed from below and the disks are placed in a regular grid with random initial velocities. The system is then allowed to relax, under the influence of gravity, up to the moment where the maximum speed detected is some small fraction of  $v_0$ . At this point a hole of length  $d_h$  appears at the center of the bottom line, and disks start pouring out of the silo. These falling disks are followed until their centers are a distance  $\approx 1.3 d_{ave}$  below the bottom, and at this moment they are eliminated from the exiting flow and re-injected on top of the system, at a distance of  $5.5 d_{ave}$  from the surface of the dense aggregate of disks. In re-injection the disks preserve their  $z$  velocity, but their horizontal velocity is set to zero. The re-injection point is chosen so as to keep the top of the material roughly flat. With these conditions the observed flow is of a mixed type, neither massic nor funnel-like; the disks close to the center of the silo fall faster than those close to the walls, but the speed difference is not too large.

Occasionally, an arch is formed above the exiting hole and the flow stops. Given that we are including static friction, these arcs are not always convex. This jammed state is detected by checking that both (1) the *maximum* speed in the system is less than  $v_0/c$ , and (2) no disk has exited the silo in a time longer than  $c t_0$ . It has been found that for the purposes of this simulation  $c = 8$  is adequate. Once these two conditions are fulfilled, the silo receives a tap given by vertical displacement  $z_{tap} = A \sin(2\pi\nu t)$ , applied for half a period. Here we have used  $A = 0.6$  and  $\nu = 8.0$ . In most cases this tap is enough to break the arch or arches that are blocking the flow; however, given that the tap moves the whole material in parallel, it does occasionally happen that there is not enough rearrangement of the disks as to break the blockage. It is possible therefore with this unjamming protocol to get *null avalanches*, which are time intervals between two taps where no material flows out of the silo. These null avalanches are highly correlated among themselves, in the sense that, for small openings, they tend to appear next to each other in the time record. This type of events have also appeared in the experiments reported by Zuriguel [18]. Null avalanches are common for very small hole sizes, less so for larger openings.

## III. RESULTS

The simulation has been carried on for hole sizes from 1.70 to 2.25, in steps of 0.05, corresponding to hole/particle ratios  $R$  from 3.4 to 4.5 in steps of 0.1. In all cases we have performed several runs starting from different grain

configurations. For each size we have obtained at least 1000 avalanches. For all hole/particle ratios the distributions of avalanches  $n(s)$  show basically an exponential form, except for a spike at  $s = 0$  (null avalanches), and a weak dip for small  $s$  (see Fig. (1)). These two characteristics are probably a peculiarity of the method used here to unjam the silo. It should be noticed that the decrease in  $n(s)$  found for small  $s$  is not as pronounced as the one reported from the experiments [10]. As for the probability of finding null avalanches, it goes from a maximum of 0.20 at  $R = 3.4$  to a minimum of 0.019 at  $R = 4.4$ , but there is not enough statistics as to be able to predict their presence or absence for larger values of  $R$ .

To avoid having to fix a bin size in the histograms, we have used the normalized cumulative distribution

$$w(s) = \sum_{s'=s}^{\infty} n(s') / \sum_{s'=1}^{\infty} n(s'),$$

that is, we count the number of avalanches with  $s$  or more disks. Notice that in this measure we are leaving out the null avalanches. For a properly normalized exponential distribution the normalized cumulative happens to be identical to the distribution itself. Fig. (2) shows the cumulative avalanche distribution for  $R = 3.6$ . Given the claim that the distribution is exponential for all cases, it should be possible to scale  $s$  to obtain a collapse of all cumulatives. This is shown in Fig. (3). Even so, it should be remembered that the distribution is not a perfect exponential, due to the smaller probabilities found for very small avalanches. This effect is almost imperceptible in the cumulatives.

The main question that remains to elucidate is the behavior of the average avalanche  $\langle s \rangle$  with respect to the hole/particle ratio. As intuitively expected, one finds a rapidly growing curve. This growth can be interpreted as evidence of an exponential divergence of the form  $\langle s(R) \rangle \approx s_0 \exp(R/R_0)$ , meaning that there is a non-zero probability of jamming for any size of the exit hole, even if for large openings the typical avalanche becomes astronomically large. However, as pointed out in [9], it is also possible to obtain a good fit to a power law of the form  $\langle s(R) \rangle \approx s_0/(R_c - R)^\gamma$ .

Trying both types of fit for the results of this simulation, we find that the the avalanche averages have a better fit to the power-law expression, with  $R_c = 6.7 \pm 0.4$  and  $\gamma = 8.16 \pm 1.10$ . For this fit the  $\chi^2/\text{dof}$  is 0.41. For the exponential fit we get  $R_0 = 0.768$  and a  $\chi^2/\text{dof}$  of 1.61. The two different fits are shown in the semi-log graph given in Fig. (4).

It is then clear that a power-law divergence is favored over an exponential behavior; it is also clear, however, that with the available data the difference between the two fits is not really large enough as to allow for a definite conclusion. Moreover, some other types of divergence have been hinted at, like an essential singularity given by  $\langle s \rangle \approx s_0 \exp[1/(R_c - R)]$ . Still, the results agree with the most extensive experiments performed at this time, and therefore adds support the existence of criticality in the jamming of silos. This leaves open the more fundamental question about the origin of the correlations that may lead to critical behavior in this type of phenomena.

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[\*] Electronic address: gperez@mda.cinvestav.mx

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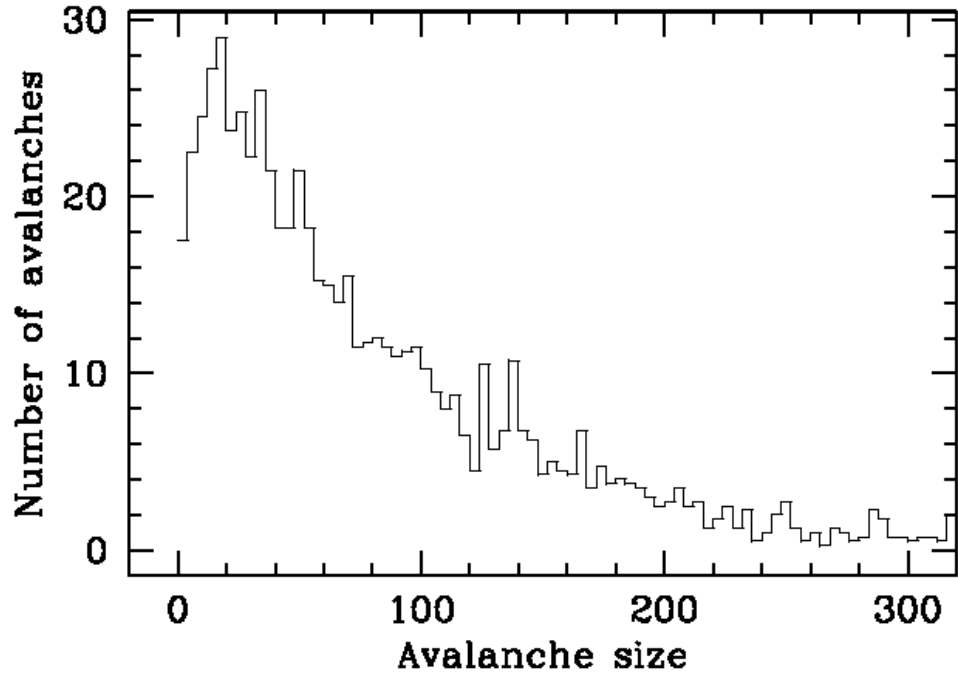


FIG. 1: Histogram of avalanches for  $R = 3.6$ . Bin size has been set to 4, and null avalanches have not been included.

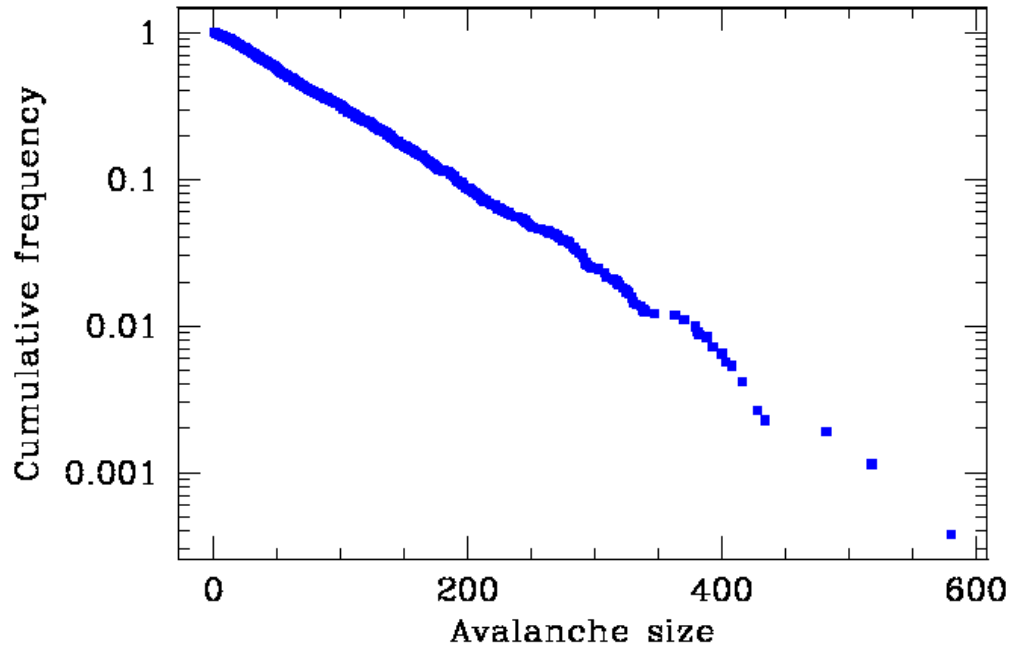


FIG. 2: Cumulative curve  $w(s)$  for  $R = 3.6$ . Null avalanches are excluded.

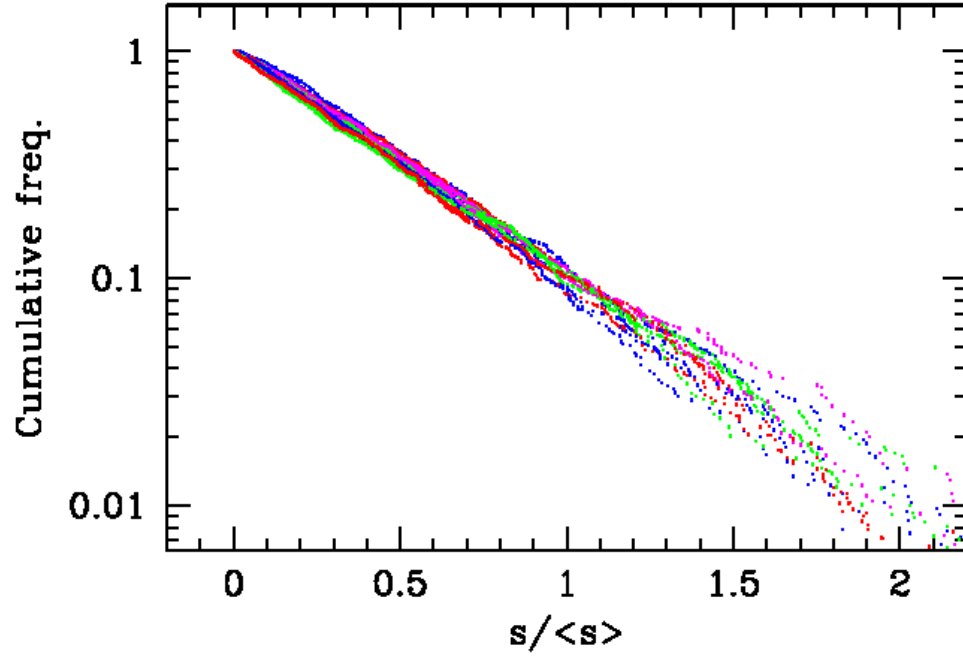


FIG. 3: Collapse of the cumulatives  $w(s)$  for all values of  $R$  considered. The regions of non-exponential behavior found for small  $s$  have some incidence over the observed dispersion, since in all cases the normalization is  $w(1) = 1$ .

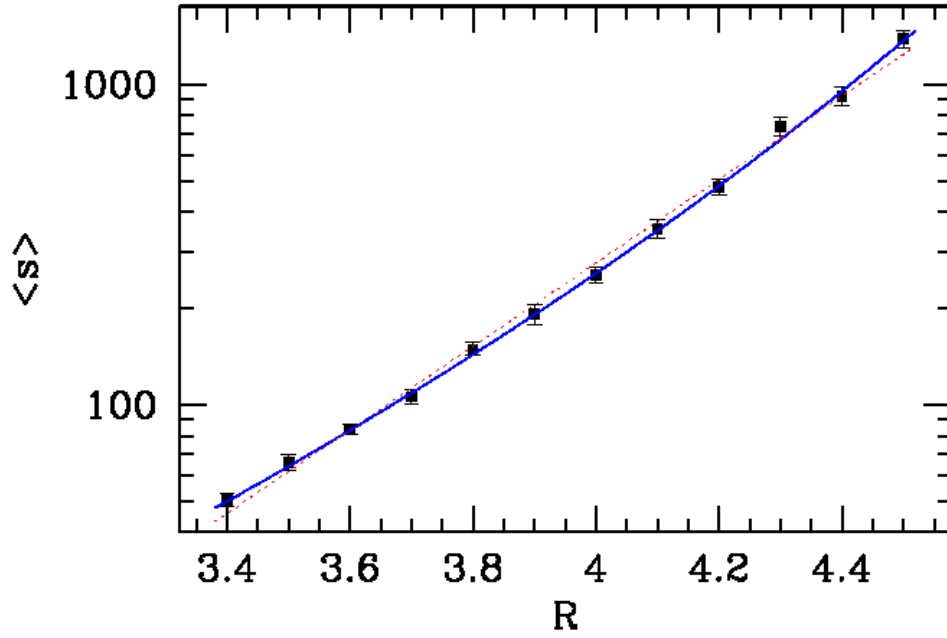


FIG. 4: Scaling of the average avalanche  $\langle s \rangle$  against the exit-hole/disk ratio  $R$ . The continuous line give the power-law fit, with  $R_c = 6.7$  and  $\gamma = 8.16$ . The dotted line shows the best linear fit, which corresponds to an exponential growth for the averages.